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5D Dirac Equation in Induced-Matter Theory

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According to an induced-matter approach, Liu and Wesson obtained the rest mass of a typical particle from the reduction of a 5D Klein–Gordon equation to a 4D one. Introducing an extra-dimension momentum operator identified with the rest mass eigenvalue operator, we consider a way to generalize the 4D Dirac equation to 5D. An analogous normal Dirac equation is gained when the generalization reduces to 4D. We find the rest mass of a particle in curved space varies with spacetime coordinates and check this for the case of exact solitonic and cosmological solution of the 5D vacuum gravitational field equations.

KEY WORDS: induced-matter theory; Kaluza–Klein theory; spacetime-matter theory; Dirac equation.

1. INTRODUCTION

In modern Kaluza–Klein theories, there was a 5D spacetime-matter theory proposed by Wesson dozens of years ago (Wesson, 1984). The theory connects the extra dimension, namely the fifth dimension, with the rest mass of a typical particle, and thereby matter is brought into geometric system. The remarkable characteristic of the approach is that the rest mass of typical particle is generally variable, which embodies Mach's principle in a sense (Liu and Mashhoon, 1995; Ma, 1990b).

The question that Wesson thought a lot about was how to generalize Einstein's equation to 5D space, that is, how to define the 5D energy–momentum tensor. At last Wesson believed the 5D energy–momentum tensor identically equals zero, that is, the 5D gravitational field equation is

$$^{(5)}R_{MN} = 0 \tag{1}$$

where index $M, N = 0, 1, 2, 3, 4$. Reducing to 4D returns it to the normal

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Einstein's equation

$$R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2}T^{\rho}\rho g_{\mu\nu}\right) \tag{2}$$

where index $\mu, \nu = 0, 1, 2, 3$. The effective 4D energy–momentum tensor $T_{\mu\nu}$ in matter field is obtained from the extra dimension by the following correspondence

$$T_{\mu\nu} = \frac{1}{8\pi G}\left[{}^{(5)}R_{\mu\nu} - \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\right] \tag{3}$$

Wesson called this approach the induced-matter theory (Wesson, 1990, 1999). Later Wesson and his Collaborators tried their best to carry it out. Recently, adhering to the induced-matter idea, Liu and Wesson generalized the Klein–Gordon equation to the 5D curved space in following form (Liu and Wesson, 2000):

$${}^{(5)}g^{MN}\phi_{;M;N} = 0 \tag{4}$$

When eq. (4) reduces to 4D, the part relating to the extra dimension is regarded as the effective mass term.

Seeing that conformal transformation preserves the causal structure of spacetime, the conformally invariant form of the field equation is worth pursuing. We further generalize eq. (4) from the minimally coupled case to the conformally invariant form

$${}^{(5)}g^{MN}\phi_{;M;N} + \frac{1}{6}{}^{(5)}R\phi = 0 \tag{5}$$

Furthermore, we check the rest mass of a typical particle in some typical gravitational fields and obtain more ideal results of the variable induced rest mass (Guo and Ma, 2001).

As is known to all, any new theory must be overall tested, modified and perfected from proposal to acceptance or denial. Many authors keep investigating the Kaluza–Klein type variable-gravity theory, namely, spacetime-matter theory (Liu and Wesson, 1994). Ma once proposed a new physical interpretation of extra-dimension subspace and a suggestion about the 5D energy–momentum tensor (Ma, 1990a, 1991). Macías discussed the Dirac equation using the Ma interpretation (Macías *et al.*, 1993). In accordance with the induced-matter idea we propose a generalization of the usual Dirac equation and discuss its properties. Because the wave function of the Dirac field has multiple components, we in principle encounter some new problems different from the Klein–Gordon equation. How to resolve these problems is one of the concerns of the present paper.

2. DIRAC EQUATION IN FIVE-DIMENSION FLAT SPACETIME

The conventional covariant form of Dirac equation in 4D spacetime is

$$i\gamma^{\mu}\psi_{,\mu} - m_0\psi = 0 \tag{6}$$

Here we choose an appropriate unit $c = \hbar = 1$. Dirac matrices γ^μ satisfies the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{7}$$

where $\eta^{\mu\nu} = \text{diagonal}(+1, -1, -1, -1)$. We choose Dirac matrices $\{\gamma^\mu\}$ as

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{8}$$

where index $i = 1, 2, 3$. I is an identical matrix. σ_i is the Pauli matrix. According to the induced-matter theory, the generalization of Dirac equation in 5D flat space is simply

$$^{(5)}\gamma^M \psi_{,M} = 0 \tag{9}$$

$$\{^{(5)}\gamma^M, ^{(5)}\gamma^N\} = 2^{(5)}\eta^{MN} \tag{10}$$

$$^{(5)}\eta^{MN} = \text{diagonal}(+1, -1, -1, -1, -1) \tag{11}$$

Here we still choose $^{(5)}\gamma^\mu$ as Eq. (8) and choose $^{(5)}\gamma^4$ as

$$^{(5)}\gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{12}$$

However, if we directly imitate the method of dealing with the scalar field in Ref.6 and Ref.7, suppose $\psi(x^M) = \psi(x^\mu)\, e^{im_0x^4}$, reduce Eq. (9) to 4D, we obtain

$$i^{(5)}\gamma^\mu \psi_{,\mu} - m_0^{(5)}\gamma^4\psi = 0 \tag{13}$$

This isn't the conventional 4D Dirac equation (6). This seems a difficulty. However, it is easy to overcome the difficulty, because it is not substantial. For example, we identify the extra-dimension momentum component operator $\hat{P}^4 \equiv \eta^{44} i \frac{\partial}{\partial x^4}$ with the rest mass operator. Considering a free Dirac particle, its wave function is

$$\psi = u(p_i, m_0)\, e^{i(p_i x^i + m_0 x^4 - Et)} \tag{14}$$

Substituting Eq. (14) in Eq. (13), we have

$$\begin{pmatrix} -EI & \vec{p} \cdot \vec{\sigma} + im_0 I \\ \vec{p} \cdot \vec{\sigma} - im_0 I & -EI \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0 \tag{15}$$

where $\chi_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $\chi_2 = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$. The essential condition of Eq. (15) having solution is that the coefficient determinant on the left-hand side is zero, which is

$$\begin{vmatrix} -IE & \hat{\vec{p}} \cdot \vec{\sigma} + im_0 I \\ \hat{\vec{p}} \cdot \vec{\sigma} + im_0 I & -EI \end{vmatrix} = 0 \tag{16}$$

It gives

$$E_{\pm} = \pm\sqrt{P^2 + m_0^2} \tag{17}$$

where E_+, E_- is positive energy and negative energy, respectively. One solves the spin eigenvalue equation to get the further solution. Hence we find the same solution as the solution of Eq. (6) for a free Dirac particle.

In fact, Eq. (13) on both hand sides multiplied left by ${}^{(5)}\gamma^0$ becomes

$$i\frac{\partial\psi}{\partial t} + i\alpha^i\frac{\partial\psi}{\partial x^i} - m_0\beta\psi = 0 \tag{18}$$

Here $\alpha^i = {}^{(5)}\gamma^{0(5)}\gamma^1 = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$, $\beta = {}^{(5)}\gamma^{0(5)}\gamma^4 = i\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, which satisfy relations

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}, \quad \{\alpha^i, \beta\} = 0, \quad \beta^2 = (\alpha^i)^2 = I$$

Therefore, Eq. (18) is the normal form of Dirac equation, that is, equivalent form of Eq. (6).

Certainly both hand sides of Eq. (13) multiplied left by $(-{}^{(5)}\gamma^4)$ gives

$$\gamma^\mu = -{}^{(5)}\gamma^{4(5)}\gamma^\mu \tag{19}$$

or, equivalently,

$$\gamma^0 = -{}^{(5)}\gamma^{4(5)}\gamma^0 = i\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \gamma^i = -{}^{(5)}\gamma^{4(5)}\gamma^i = i\begin{pmatrix} -\sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

here γ^μ satisfy the anticommunication relations (7). Equation (13) becomes

$$i\gamma^\mu\psi_{,\mu} - m_0\psi = 0 \tag{20}$$

If we define Dirac matrix γ^μ in the 4D Dirac equation as (19), Eq. (20) is another spinor expression of Eq. (6).

We conclude this section by stressing that it is necessary to introduce an extra-dimension momentum operator, namely, rest-mass operator $\hat{p}^4 \equiv \eta^{44} i \frac{\partial}{\partial x^4}$ for two other important reasons. First, it is required by covariance of the theory. Second, existence of m_0 is natural by introducing the rest-mass operator. In our present opinions, m_0 is nonzero eigenvalue of the rest-mass operator and different eigenvalues represent different rest-mass particles. That is to say, existence of $\hat{p}^4$

is natural for 5D generalization of the theory. Therefore, existence of its eigenvalue is natural too. By identifying $\hat{p}^4$ with rest-mass operator, an extra-dimension momentum has an explicit physical interpretation. Meanwhile, a particle obtains a 4D effective rest mass.

3. DIRAC EQUATION IN CURVED SPACETIME

Let recall the case in 4D spacetime. Considering that gravity affects on a Dirac spinor particle, that is, the Dirac equation is generalized to the curved spacetime, we must introduce vierbein field (Birrell and Davies, 1982; Wald, 1984)

$$V_{\mu}^{\hat{\alpha}}(X) = \left(\frac{\partial \zeta_X^{\hat{\alpha}}(x)}{\partial x^{\mu}}\right)_{x=X} \tag{21}$$

Where $\zeta_X^{\hat{\alpha}}(x)$ is the local coordinate at X and index $\hat{\alpha}$ is Lorentz index that is raised or lowed by η. The metric tensor is related to $\eta_{\hat{\alpha}\hat{\beta}}$ by

$$g_{\mu\nu} V_{\hat{\alpha}}^{\mu} V_{\hat{\beta}}^{\nu} = \eta_{\hat{\alpha}\hat{\beta}} \quad g_{\mu\nu} = V_{\mu}^{\hat{\alpha}} V_{\nu}^{\hat{\beta}} \eta_{\hat{\alpha}\hat{\beta}} \tag{22}$$

The spin covariant derivative is defined by

$$\nabla_{\mu}\psi = (\partial_{\mu} + \Gamma_{\mu})\psi \tag{23}$$

Where Γ_{μ} are the spin connections

$$\Gamma_{\mu} = \frac{1}{2} V_{\hat{\alpha}}^{\nu} V_{\hat{\beta}\nu;\mu} \sigma^{\hat{\alpha}\hat{\beta}} \tag{24}$$

Here $V_{\hat{\beta}\nu;\mu} = V_{\hat{\beta}\nu,\mu} - V_{\hat{\beta}\lambda}\Gamma_{\nu\mu}^{\lambda}$ depends on the Christoffel symbols, and $\sigma^{\hat{\alpha}\hat{\beta}} = \frac{1}{4}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]$ is called the Lorentz group generators. The generalization of Dirac equation in curved spacetime is

$$i\gamma^{\mu}\nabla_{\mu}\psi - m\psi = 0 \tag{25}$$

where γ^{μ} satisfy the anticommuniation relations

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \tag{26}$$

According to the idea of induced-matter theory, the generalization of Dirac equation in 5D curved space is

$${}^{(5)}\gamma^{M(5)}\nabla_{M}\psi = 0 \tag{27}$$

This involves the Dirac matrix, vierbein, spin connections, and spin covariant derivative, which are the direct generalizations of the 4D case.

4. EFFECTIVE MASS OF A DIRAC PARTICLE IN 5D THEORY

When the Dirac equation (27) in 5D curved space reduce to the 4D form, that is, compared with Eq. (25) in 4D curved space, an effective mass m_{eff} is yielded, which is called induced mass. We should point out that for the case in 5D flat space the reducing procedure is completed by solving operator $\hat{p}^4$ eigenvalue equation (see Section 2). The induced mass is the eigenvalue of operator $\hat{p}^4$, which is constant. However, for the case of 5D curved space, the rest-mass terms are not determined only by operator $\hat{p}^4$, and the momentum is not constant because of the effects of gravity.

In many Kaluza–Klein-type theories, for the sake of simplicity ${}^{(5)}g_{4\mu} = 0$ is usually chosen, so ${}^{(5)}g_{\mu\nu} = g_{\mu\nu}$. The 5D line element is

$$ {}^{(5)}dS^2 = {}^{(5)}g_{MN}\, dx^M\, dx^N = g_{\mu\nu}\, dx^\mu\, dx^\nu - \phi^2\, dl^2 \tag{28} $$

where $l \equiv x^4$. In the horizontal lift basis the vierbein is block diagonal,

$$ {}^{(5)}V_M^{\hat{A}} = \begin{pmatrix} V_\mu^{\hat{\alpha}} & 0 \\ 0 & \phi \end{pmatrix} \tag{29} $$

The nonzero 5D Christoffel symbols are

$$ \begin{aligned} {}^{(5)}\Gamma^\mu_{\nu\lambda} &= \Gamma^\mu_{\nu\lambda}, \quad {}^{(5)}\Gamma^\mu_{\nu 4} = \frac{1}{2} g^{\mu\lambda} \overset{*}{g}_{\nu\lambda} \\ {}^{(5)}\Gamma^\mu_{44} &= \phi g^{\mu\nu} \phi_{,\nu}, \quad {}^{(5)}\Gamma^4_{\mu\nu} = \frac{1}{2} \phi^{-2} \overset{*}{g}_{\mu\nu} \\ {}^{(5)}\Gamma^4_{4\mu} &= \phi^{-1}\phi_{,\mu}, \quad {}^{(4)}\Gamma^4_{44} = \phi^{-1}\overset{*}{\phi} \end{aligned} \tag{30} $$

where partial derivatives with respect to l are denoted by an asterisk. The spin connections are

$$ \begin{aligned} {}^{(5)}\Gamma_\mu &= \Gamma_\mu \frac{1}{2} \phi^{-1} \overset{*}{g}_{\mu\nu} V^\nu_{\hat{\alpha}} \sigma^{\hat{\alpha}4} \\ {}^{(5)}\Gamma_4 &= \frac{1}{2} V^\mu_{\hat{\alpha}} \sigma^{\hat{\alpha}\hat{\beta}} \left(\overset{*}{V}_{\hat{\beta}\mu} - \frac{1}{2} \overset{*}{g}_{\mu\nu} V^\nu_{\hat{\beta}} \right) - \phi_{,\mu} V^\mu_{\hat{\alpha}} \sigma^{\hat{\alpha}\hat{4}} \end{aligned} \tag{31} $$

From the Dirac equation (27) we have

$$ \begin{aligned} 0 &= i^{(5)}\gamma^{M(5)}\nabla_M \psi = i^{(5)}\gamma^{\mu(5)}\nabla_\mu \psi + i^{(5)}\gamma^{4(5)}\nabla_4 \psi \\ &= i^{(5)}\gamma^\mu \nabla_\mu \psi + \frac{1}{2} i^{(5)}\gamma^\mu \phi^{-1} \overset{*}{g}_{\mu\nu} V^\nu_{\hat{\alpha}} \sigma^{\hat{\alpha}\hat{4}} \psi + i^{(5)}\gamma^4 \overset{*}{\Psi} \\ &\quad + i^{(5)}\gamma^4 V^\mu_{\hat{\alpha}} \left[\frac{1}{2} \sigma^{\hat{\alpha}\hat{\beta}} \left(\overset{*}{V}_{\hat{\beta}\mu} - \frac{1}{2} \overset{*}{g}_{\mu\nu} V^\nu_{\hat{\alpha}} \right) - \phi_{,\mu} \sigma^{\hat{\alpha}\hat{4}} \right] \psi \end{aligned} \tag{32} $$

Before discussing the effective rest mass m_{eff}, we note the following: (i) the momentum operator in cured space should be the general covariant gradient operator (multiplied by i), namely, $i^{(5)}\nabla_M$. The eigenvalue of $-i\frac{\partial}{\partial l}$ is not the effective rest mass of a Dirac particle; (ii) we see that ${}^{(5)}g_{\mu 4} = 0$, ${}^{(5)}g_{44} = \phi^2$, and ${}^{(5)}g_{\mu\nu} = g_{\mu\nu}$, so the generalization of Eq. (19) is

$$\gamma^\mu = -\frac{1}{\sqrt{-g_{44}}}{}^{(5)}\gamma^{4(5)}\gamma^\mu = -\phi^{-1(5)}\gamma^{4(5)}\gamma^\mu \tag{33}$$

The defined γ^μ satisfy the anticommunication relations (26). Multiplying Eq. (32) on both hand sides left by $\left(-\frac{1}{\sqrt{-g_{44}}}{}^{(5)}\gamma^4\right)$ with Eq. (33) and setting $\psi(x^M) = \psi(x^\mu)\, e^{im_0 l}$, we obtain

$$\begin{aligned} 0 &= i\gamma^\mu\nabla_\mu\psi + i\psi + \frac{1}{2}i\gamma^\mu\phi^{-1}\overset{*}{g}_{\mu\nu}V^\nu_{\hat\alpha}\sigma^{\hat\alpha\hat 4}\psi \\ &\quad + iV^\mu_{\hat\alpha}\left[\frac{1}{2}\sigma^{\hat\alpha\hat\beta}\left(\overset{*}{V}_{\hat\beta\mu} - \frac{1}{2}\overset{*}{g}_{\mu\nu}V^\nu_{\hat\alpha}\right) + \phi_{,\mu}\sigma^{\hat\alpha\hat 4}\right]\psi \\ &= i\gamma^\mu\nabla_\mu\psi - \left[m_0 - \frac{1}{2}i\gamma^\mu\phi^{-1}\overset{*}{g}_{\mu\nu}V^\nu_{\hat\alpha}\sigma^{\hat\alpha\hat 4}\right. \\ &\quad \left. -\frac{1}{2}iV^\mu_{\hat\alpha}\sigma^{\hat\alpha\hat\beta}\left(\overset{*}{V}_{\hat\beta\mu} - \frac{1}{2}\overset{*}{g}_{\mu\nu}V^\nu_{\hat\alpha}\right) - iV^\mu_{\hat\alpha}\phi_{,\mu}\sigma^{\hat\alpha\hat 4}\right]\psi \end{aligned} \tag{34}$$

Comparing Eq. (34) with Eq. (25) gives

$$\begin{aligned} m_{\text{eff}}\psi &= \left[m_0 - \frac{1}{2}i\gamma^\mu\phi^{-1}\overset{*}{g}_{\mu\nu}V^\nu_{\hat\alpha}\sigma^{\hat\alpha\hat 4} - \frac{1}{2}iV^\mu_{\hat\alpha}\sigma^{\hat\alpha\hat\beta}\left(\overset{*}{V}_{\hat\beta\mu}\right.\right. \\ &\quad \left.\left. -\frac{1}{2}\overset{*}{g}_{\mu\nu}V^\nu_{\hat\alpha}\right) - iV^\mu_{\hat\alpha}\phi_{,\mu}\sigma^{\hat\alpha\hat 4}\right]\psi \end{aligned} \tag{35}$$

We find that in general m_{eff} is variable with spacetime coordinates, which is expected by the theory.

5. APPLICATION IN SPECIFIC GRAVITY

In this section we consider two specific examples. First, consider the static, spherically-symmetric gravity. The line element is (Dauidson and Owen, 1985; Gross and Perry, 1983)

$$\begin{aligned} dS^2 &= A^a\,dt^2 - A^{-a-b}\,dr^2 - A^{1-a-b}r^2\,d\Omega^2 - A^b\,dl^2 \\ A(r) &\equiv 1 - 2GM/r \quad 1 = a^2 + ab + b^2 \end{aligned} \tag{36}$$

We choose the vierbein as

$$V^{\hat A}_M = diagonal\left(A^{\frac{a}{2}}, A^{\frac{-(a+b)}{2}}, A^{\frac{1-(a+b)}{2}}r, A^{\frac{1-(a+b)}{2}}r\,\sin\theta, A^{\frac{b}{2}}\right) \tag{37}$$

Expression (35) become

$$\begin{pmatrix} (m_{\text{eff}} - m_0)I - \frac{1}{2}\frac{bGM}{r^2}A^{\left(b-\frac{a}{2}-1\right)}\sigma_1 & 0 \\ 0 & (m_{\text{eff}} - m_0)I + \frac{1}{2}\frac{bGM}{r^2}A^{\left(b-\frac{a}{2}-1\right)}\sigma_1 \end{pmatrix} \times \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0 \tag{38}$$

The essential condition of Eq. (38) having solution is that the coefficient determinant on the left-hand side is zero, which is

$$m_{\text{eff}} - m_0 \pm \frac{1}{2}\frac{bGM}{r^2}A^{\left(b-\frac{a}{2}-1\right)} \tag{39}$$

Ref. 7 fixes the values of a and b that $b \approx 0.002$, $a \approx 0.999$. From Eq. (39) returned to the usual unit we get

$$m_{\text{eff}} = m_0 \pm 0.001\frac{GM\hbar}{r^2c^3}\left(1 - \frac{GM}{c^2r}\right)^{-1.498} \tag{40}$$

As an illustration, let us consider a electron passing the Sun with dimensionless potential $\frac{GM_\oplus}{R_\oplus c^2} = 2.12 \times 10^{-6}$, $r = R_\oplus = 6.96 \times 10^{10}$ cm and mass of a electron $m_0 = m_{\text{e}} = 9.11 \times 10^{-28}$ g. Using Eq. (40) we obtain

$$m_{\text{eff}} = m_0(1 \pm 1.17 \times 10^{-30}) \tag{41}$$

Though the result cannot be tested exactly by experiment, we find at least no conflict with experience, which has great theoretical significance, that is, embodies Mach's principle.

Consider the case of cosmology and choose the line element as

$${}^{(5)}dS^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-r^2} + r^2\,d\Omega^2\right) - f^2(t)\,dl^2 \tag{42}$$

Obviously, ${}^{(5)}g_{\mu\nu} = g_{\mu\nu}$ is the Robertson–Walker metric in the 4D close ($k = +1$) universe model. The nonzero components of the Ricci tensor are

$$\begin{aligned} {}^{(5)}R_{00} &= 3\frac{\ddot{R}}{R} + \frac{\ddot{f}}{f} = R_{00} + \frac{\ddot{f}}{f} \\ {}^{(5)}R_{11} &= -\frac{1}{1-r^2}(R\ddot{R} + 2\dot{R}^2 + 2) - \frac{1}{1-r^2}R\dot{R}\frac{\dot{f}}{f} \\ &= R_{11} - \frac{1}{1-r^2}R\dot{R}\frac{\dot{f}}{f} \\ {}^{(5)}R_{22} &= r^2(1-r^2){}^{(5)}R_{11} = R_{22} - r^2R\dot{R}\frac{\dot{f}}{f} \end{aligned}$$

$$
\begin{aligned}
{}^{(5)}R_{33} &= \sin^2\theta\, {}^{(5)}R_{22} = R_{33} - r^2 \sin^2\theta\, R\dot{R}\frac{\dot{f}}{f} \\
{}^{(5)}R_{44} &= -f\ddot{f} - 3f\dot{f}\frac{\dot{R}}{R}
\end{aligned} \tag{43}
$$

Here overdots denote partial derivatives with respect to t. The 5D gravitational field equation ${}^{(5)}R_{MN} = 0$ gives

$$
\begin{aligned}
&\ddot{f} + 3f\frac{\ddot{R}}{R} = 0 \\
&\ddot{f} + 3f\frac{\dot{R}}{R} = 0 \\
&\dot{f} + f\left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}}{R} + 2\frac{1}{R\dot{R}}\right) = 0
\end{aligned} \tag{44}
$$

The first two equations in Eq. (44) give

$$
f = \dot{R} \tag{45}
$$

The last equation in Eq. (44) with Eq. (45) becomes

$$
R\ddot{R} + \dot{R}^2 + 1 = 0 \tag{46}
$$

In the 4D close Robertson–Walker universe model, if the equation of state is $p = \rho/3$, the dynamical equation gives just Eq. (46). We have

$$
\ddot{R} = -\frac{8\pi G}{3}\rho R = -\frac{8\pi G}{3}\rho_0 R_0^4 \frac{1}{R^3} \tag{47}
$$

where ρ_0 and R_0 are the present values of the matter density and the scale factor, respectively. We find equation Eq. (45) equivalently gives the equation of state $p = \frac{1}{3}\rho$

The solution of eq. Eq. (47) is

$$
R(t) = \frac{1}{\sqrt{\alpha}}\sqrt{A^2 - (A - \alpha t)^2} \tag{48}
$$

where $A = \sqrt{\frac{8\pi G}{3}\rho_0}R_0^2$. Using $\left.\frac{\dot{R}}{R}\right|_{t_0} = H_0$, $\frac{8\pi G}{3}\rho_0 > H_0^2$, and $\frac{8\pi G}{3}\rho_0 > H_0^2$ (because of close universe) this gives the integral constant.

$$
\alpha = \left(\frac{8\pi G}{3}\rho_0 - H_0^2\right)R_0^2 \tag{49}
$$

We further calculate f^2

$$
f^2 = \frac{\alpha(A - \alpha t)^2}{A^2 - (A - \alpha t)^2} \tag{50}
$$

The vierbein is chosen as

$$V_M^{\hat{A}} = \text{diagonal}\left(1, \frac{R}{\sqrt{1-r^2}}, Rr, Rr\ \sin\theta, f\right) \tag{51}$$

Equation (35) becomes

$$\begin{pmatrix} (m_{\text{eff}} - m_0 + \frac{1}{2}\ddot{R})I & 0 \\ 0 & (m_{\text{eff}} - m_0 - \frac{1}{2}\ddot{R})I \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0 \tag{52}$$

The essential condition of Eq. (52) having solution is that the coefficient determinant on the left-hand side is zero, which yields

$$m_{\text{eff}} = m_0 \pm \frac{1}{2}\ddot{R} \tag{53}$$

Returning to the usual unit, we rewrite expression (53) as

$$m_{\text{eff}} = m_0 \pm \frac{1}{2}\frac{\hbar}{c^2}\ddot{R} \tag{54}$$

Though the cosmological solution is not suitable for the present universe, we can give a rough estimate using the present values. With the present value of the decelerating parameter $q_0 \approx 1$ and the present value of the Hubble constant $H_0 \approx (4.11 \times 10^{17}\text{s})^{-1}$, then $R_0 \approx H_0^{-1}$ and $8\pi G\rho_0/3 \approx 2H_0^2$, expression (47) gives

$$\ddot{R}_0 \approx \frac{8\pi G}{3}\rho_0 R_0 \approx -2H_0 \tag{55}$$

With the electron mass $m_e \approx 9.11 \times 10^{-28}$ g expression (54) gives

$$m_{\text{eff}} = m_e(1 \pm 1.5 \times 10^{-30}) \tag{56}$$

We find the result is reasonable again.

6. CONCLUSION

According to the induced-matter idea proposed by Wesson and Liu, we establish the conformally invariant 5D Dirac equation on the basis of the conformally invariant 5D Klein–Gordon equation. Because the wave function of Dirac field is multiple component matrix form, the method of constructing the Dirac equation doesn't simply imitate that of the Klein–Gordon equation. In 5D flat space we firstly identify extra-dimension momentum operator with the rest-mass operator and derive the mass terms when the 5D massless Dirac equation reduces to 4D by solving momentum eigenvalue problem. We find the proper relation (33) between 5D Dirac matrix and 4D Dirac matrix, generalize the Dirac equation to the cured space, and obtain the general expression of the effective induced matter. We check our results for the two specific cases of the static, spherically-symmetric gravity, and the 5D cosmological solution, which we obtain corresponding to the 4D close Roberston–Walker cosmological model. The result is satisfactory.

To sum up, according to the above method of establishing the Dirac equation, there are some characters as follows: First, it accords with the induced-matter theory and was once Einstein's dream (Wesson, 1999). Second, the equation is conformally invariant. Third, as for 5D theory it gives the extra-dimension momentum operator and the specific physical interpretation of eigenvalue. Fourth, the effective induced matter is generally related to spacetime geometry that depends on the distribution of matter in the universe. Consequently mass depends on the distribution of matter in the universe, which embodied the spirit of Mach principle. Fifth, we use the typical gravity to examine that the result has no conflict with observations.

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